Max Flows 2: Edmonds-Karp

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As we saw, F-F is a pseudopolynomial-time algorithm (assuming integral capacities), takes time $O(|f_{max}|)$.
To get a poly time algo, we will always pick the shortest s-t path in the residual graph G_f. This is known as the Edmonds - Karp algorithm.

<u>Theorem:</u> E-K algorithm terminates in $O(mn)$ iterations.

For the proof: will use f_i to denote flow after i^{th} iteration ($f_0 = 0$)
G_{f_i} is the usual residual graph
$dist_i(v)$ is the shortest-path distance of v from s in G_{f_i}

Thus, if v is reachable from s in G_{f_i}, then $dist_i(v) \le n-1$
Else $dist_i(v) = +\infty$

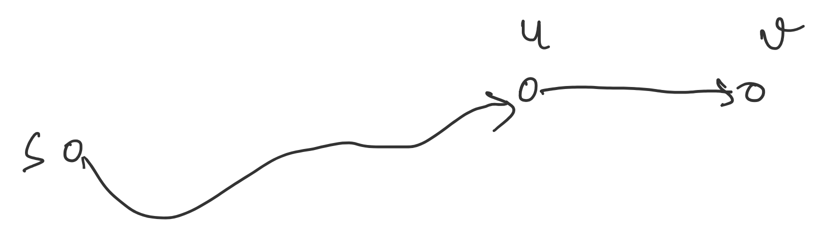
<u>Also</u>: note that in each iteration of E-K / F-F, if $e = (u,v) \in P$ & $c^{f}_{uv} = \delta$, then (u,v) is not present in the next residual graph.
Thus at least one edge disappears in each iteration of E-K.
(but other edges could appear)

<u>Claim 1:</u> $\forall v,\ dist_{i+1}(v) \ge dist_i(v)$

<u>Claim 2:</u> For any $e = (u,v) \in E$, # of times either (u,v) re-appears or (v,u) re-appears is $O(n)$,
i.e., $\left| \{ i : (u,v) \notin G_{f_i},\ (u,v) \in G_{f_{i+1}} \} \cup \{ i : (v,u) \notin G_{f_i},\ (v,u) \in G_{f_{i+1}} \} \right| = O(n)$

<u>Proof of Theorem:</u> Since each edge can reappear at most $O(n)$ times, it can disappear $O(n)$ times. Since in each iteration an edge must disappear, after mn iterations, the # of disappearances is mn. But after $O(mn)$ disappearances, there are no edges remaining, and hence no s-t path, and f must be a max flow ∎

<u>Proof of Claim 1:</u> For a contradiction, assume $\exists v, i$, $dist_{i+1}(v) < dist_i(v)$.
Let $i+1$ be first iteration s.t. $\exists v$, $dist_{i+1}(v) < dist_i(v)$
Of vertices for which $dist_{i+1}(v) < dist_i(v)$, let $\hat{v}$ minimize $dist_{i+1}(v)$
(Thus for any v s.t. $dist_{i+1}(v) < dist_{i+1}(\hat{v})$, $dist_{i+1}(v) \ge dist_i(v)$)



Let (u,v) be the last edge in shortest path to v in $G_{f_{i+1}}$

Thus $dist_{i+1}(u) = dist_{i+1}(v) - 1$ —— (**)
and thus $dist_{i+1}(u) \ge dist(u)$ —— (*)

<u>Case I:</u> $(u,v) \in G_{f_i}$. But then
$dist_i(v) \le dist_i(u) + 1$
$\le dist_{i+1}(u) + 1 = dist_{i+1}(v)$

<u>Case II:</u> $(u,v) \notin G_{f_i}$. Since $(u,v) \in G_{f_{i+1}}$, then (v,u) must have been on the s-t shortest-path.
Thus $dist_i(v) = dist_i(u) - 1$
$\le dist_{i+1}(u) - 1$
$= dist_{i+1}(v) - 2$ ∎

Thus if (u,v) edge reappeared, the dist of v increased by 2. And hence this edge can reappear $O(n)$ times.

(can prove Claim 2 v. similar to Case II of Claim 1)

Using appropriate data structures, can implement E-K to run in time $O(m^2 n)$.